Report on "Empirical likelihood method in statistical inference"

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Complete a nonrigorous sketch of the proof of the following theorem by solving Q1 and Q2 below.

Theorem 2.2 (Univariate ELT, in Chapter 2 of Owen's book)
Let $X_1, \ldots, X_n \in \mathbb{R}$ be independent random variables with common distribution F_0. Let $\mu_0 = \mathrm{E}(X_i)$, and suppose that $0 < \mathrm{Var}(X_i) < \infty$. Then $-2\log(\mathcal{R}(\mu_0)) \xrightarrow{d} \chi^2_{(1)} \quad (n \to \infty)$.

Sketch of the proof
The profile empirical likelihood for the true mean μ_0 is written by

$$\mathcal{R}(\mu_0) = \max_{(p_1,\ldots,p_n)} \left\{ \prod_{i=1}^n np_i \;\middle|\; \sum_{i=1}^n p_i X_i = \mu_0, p_i \geq 0, \sum_{i=1}^n p_i = 1 \right\}$$

and the maximizer $(p_1, \ldots, p_n)$ is given by

$$p_i = \frac{1}{n} \frac{1}{1 + \lambda(X_i - \mu_0)} \tag{1}$$

where the Laglange multiplier λ satisfies

$$\frac{1}{n} \sum_{i=1}^n \frac{X_i - \mu_0}{1 + \lambda(X_i - \mu_0)} = 0. \tag{2}$$

Q1. Expand equation (2) in a Taylor series about $\lambda = 0$. Using the leading terms, show that $\lambda \doteq (\bar{X} - \mu_0)/S(\mu_0)$ where $S(\mu_0) = (1/n)\sum_{i=1}^n (X_i - \mu_0)^2$.

Q2. Substitute (1) and $\lambda \doteq (\bar{X} - \mu_0)/S(\mu_0)$ from Q1 into an expression for $-2\log(\mathcal{R}(\mu_0)) = -2\log(\prod_{i=1}^n np_i)$ and show, after a Taylor approximation, that the result is nearly equal to $n(\bar{X} - \mu_0)^2/S(\mu_0)$. (Use the approximation $\log(1+x) \doteq x - (1/2)x^2$).

A $\chi^2_{(1)}$ limit is then reasonable when $\sqrt{n}(\bar{X} - \mu_0)$ is asymptotically normal with a mean 0 and a variance estimated by $S(\mu_0)$.