Lecture 9 Applications of Static Games of Incomplete Information

Good to be sold at an auction. Which auction design should be used in order to maximize expected revenue for the seller, if the bidders valuation of the good is private knowledge?

English auction: Oral bids, ascending until nobody wants to make a higher bid. The person with the highest bid gets the good and has to pay his highest bid. Most used auction form, used e.g. for art auctions.

Dutch auction: Auctioneer starts with a high price, then lowers it gradually until the first buyer accepts. This buyer gets the good and has to pay the price he accepted. Used e.g. for flower and vegetable auctions.

Sealed bid auction: each bidder makes just one bid simultaneously, and the bidder with the highest price wins. If two or more bidders make the same highest bid, then winner randomly chosen.

First price: Winner pays his own bid, i.e. the highest bid

Second price: Winner pays the highest of the *loosing* bids, i.e. the second highest bid.

The framework

Two bidders, who can make non-negative bids.

The value of the good for each bidder is private information: bidder i knows only his own valuation $v_{i.}$

Independent draws of the valuation for each bidder: "Private value auction"

Example of private value: art object bought by collector

Opposite: Good has the same value for everyone, but true value is unknown to the bidders: "Public value auction".

Example: mobile phone licence.

v_i drawn from uniform distribution over $[0,1]$, i.e. for $0 \leq k \leq l \leq 1$

$$prob\{k \leq v_i \leq l\} = l - k$$

Note: $prob\{v_i = c\} = 0$ for all c.

payoffs:

$$\begin{aligned} u_i &= v_i - a \text{ if bidder } i \text{ wins and pays price } a \\ u_i &= 0 \text{ otherwise.} \end{aligned}$$

1. Sealed bid auction

Bayesian game $G = \{A_1, ...A_n; T_1, ... T_n; p_1, ...p_n; u_1, ...u_n\}$

Action sets: $A_1 = A_2 = [0, \infty)$

Type spaces: $T_1 = T_2 = [0, 1]$

$\Longrightarrow$ Strategy of player i:

$$s_i : [0, 1] \rightarrow [0, \infty)$$

with $s_i(v_i)$ being $i's$ bid when $i's$ valuation is v_i

beliefs:

$i's$ belief that $j's$ valuation is between k and l is given by

$$p_i(k < v_j < l \,|\, v_i) = l - k = p_i(k < v_j < l)$$

$i's$ belief that $j's$ valuation is c is given by:

$$p_i(v_j = c) = 0$$

Reason: independent drawings from uniform distributions

Other extreme case: drawings completely aligned:

$$p_i(k \leq v_j \leq l \,|\, v_i) = \begin{cases} 1 & \text{if } k \leq v_i \leq l \\ 0 & \text{else} \end{cases}$$

1.1. First price sealed bid auction

expected payoff of player i with a valuation v_i and a bid a_i, given player $j's$ strategy $s_j(v_j)$

$$
\begin{aligned}
u_i(v_i, a_i, s_j(v_j)) &= prob\{a_i > s_j(v_j)\}(v_i - a_i) + \\
&\quad \frac{1}{2} prob\left\{a_i = s_j(v_j)\right\}(v_i - a_i)
\end{aligned}
$$

Proposition: Both players choosing the strategy $s_i^*(v_i) = \frac{v_i}{2}$ is a Bayesian Nash equilibrium.

Proof:

In order to prove the proposition, we have to show that for every type (i.e. for every valuation) the proposed bid indeed maximizes the expected payoff of player i, given the strategy of player j. Take player 1 with a valuation v_1 and assume that player 2 bids $\frac{v_2}{2}$ for all possible v_2. Then

$$\begin{aligned} prob\{a_1 &> s_2(v_2)\} = 1 - prob\{a_1 \leq \frac{v_2}{2}\} \\ &= 1 - p_1\left\{2a_1 \leq v_2 \leq 1\right\} \\ &= \begin{cases} 1-(1-2a_1) = 2a_1 & \text{if } a_1 \leq \frac{1}{2} \\ 1 & \text{else} \end{cases} \end{aligned}$$

$$prob\{a_1 = s_2^*(v_2)\} = 0$$

$$\Rightarrow u_1(v_1, a_1, s_2^*(v_2)) = \begin{cases} 2a_1(v_1 - a_1) & \text{if } a_1 < \frac{1}{2} \\ (v_1 - a_1) & \text{if } a_1 \geq \frac{1}{2} \end{cases}$$

If $a_1 < \frac{1}{2}$:

$$\begin{aligned} 0 &= 2v_1 - 4a_1^* \Rightarrow \\ a_1^* &= \frac{v_1}{2} \end{aligned}$$

if $a_1 \geq \frac{1}{2}$:

$$a_1^* = \frac{1}{2}$$

Hence, $a_1^* = \frac{v_1}{2}$ is the optimal action for any v_1 against player $2's$ strategy, and therefore player $1's$ strategy $s_1^*(v_1) = \frac{v_1}{2}$. By symmetry, player $2's$ strategy is also optimal against player $1's$ strategy $\Rightarrow s_i^*(v_i) = \frac{v_i}{2}$ for both players is a Nash equilibrium■.

Revenue of the seller π_s:

Denote by w the player with the higher valuation v_w, and by l the player with the lower valuation v_l. Due to uniform distribution, the expected $v_l = \frac{v_w}{2}$. Hence, the seller's expected revenues are:

$$\pi_s = \frac{v_w}{2} = v_l$$

1.2. Second price sealed bid auction

expected payoff of player i with a valuation v_i and a bid a_i, given player $j's$ strategy $s_j(v_j)$:

$$\begin{aligned} u_i(v_i, a_i, s_j(v_j)) &= prob\{a_i > s_j(v_j)\}(v_i - s_j(v_j)) + \\ &\quad \frac{1}{2} prob\left\{a_i = s_j(v_j)\right\}\left(v_i - s_j(v_j)\right) \end{aligned}$$

Proposition: Both players choosing the strategy $s_i^*(v_i) = v_i$ is a Bayesian Nash equilibrium.

Proof: Take player 1 with a valuation v_1 and a bid $a_1^* = v_1$. We have to distinguish between 3 cases:

$a_2 > a_1^* = v_1$. In this case, any other bid a_1' with $a_1' < a_2$ would not make any difference for player 1, since he still does not win. With the alternative bid $a_1' > a_2$, player 1 wins the good, but has to pay a price of a_2 which is above his valuation of the good -1 makes a loss. Finally, for the alternative bid $a_1' = a_2$, there is a chance that player 1 wins the good and makes a loss. Hence, no profitable deviation from $a_1^* = v_1$ is possible when $a_2 > v_1$.

$a_2 < a_1^* = v_1$. In this case, any other bid a_1' with $a_1' > a_2$ would not make any difference for player 1, since he would still win and pay a_2. With the alternative bid $a_1' < a_2$, player 1 does no longer win the good and gets zero, whereas with $a_1^* = v_1$ he would get $v_1 - a_2 > 0$. Finally, for the alternative bid $a_1' = a_2$, there is a chance that player 1 does not win the good, although winning would be profitable. Again, no profitable deviation from $a_1^* = v_1$ is possible.

$a_2 = a_1^* = v_1$. In this case, any other bid a_1' with $a_1' > a_2$ would not make any difference for player 1, since he wins and pays a_2 so that his utility would still be zero. With the alternative bid $a_1' < a_2$, player 1 does no longer win the good and gets zero, too. Again, no profitable deviation from $a_1^* = v_1$ is possible.

Hence, for all possible a_2, no type of player 1 can profitably deviate from $a_1^* = v_1$. By symmetry, the same holds for player 2■.

Revenue of the seller π_s

Denote by w the player with the higher valuation v_w, and by l the player with the lower valuation v_l. Due to uniform distribution, the expected $v_w = 2v_l$. Hence, the seller's expected revenues are:

$$\pi_s = v_l = \frac{v_w}{2}$$

Revenue equivalence: The same expected revenue for the seller in the first and in the second price sealed bid auction!

Reason: For given bids, the revenues are higher in the first price auction. However, bidders bid higher in the second price auction, since the winner's payment does not depend on his own bid. Both effects exactly offset each other.

Revenue equivalence holds for any number of bidders and any distribution of values, as long as the values are drawn independently - pure private value auctions.

For n bidders:

First price sealed bid auction: $s_i^*(v_i) = \frac{(n-1)v_i}{n}$

Second price sealed bid auction: $s_i^*(v_i) = v_i$

2. English auction

As long as at least two players bid, none of the bidder can gain, if he stops before his valuation is reached. No bidders bids higher, when his valuation level is reached. No bidder goes on if the second highest valuation is overbid by the smallest possible amount, i.e. one cent. Hence, everyone's but the winner's highest bid is his valuation, and the price paid by the winner is (nearly) the second highest valuation $\Rightarrow$

The English auction generates (nearly) the same bidding behavior as the second price sealed bid auction. It leads to the same winner (bidder with the highest valuation), and the same revenues between the English and the second price sealed bid auction differ only by 1 cent.

Again, it can be shown that this equivalence holds whenever we have a pure private value auction.

3. Dutch auction

The Dutch auction is equivalent to the first price sealed auction.

Reason: When in a Dutch auction a player is the first to accept the descending price, he acts without knowing when the others would accept. Hence, accepting a certain price in the Dutch auction is equivalent as bidding the same price in the first price sealed bid auction. Furthermore, the consequences are the same: The winner pays the accepted price (i.e. his own bid). Therefore, any equilibrium in the Dutch auction has the same outcome as the any equilibrium in the first price sealed bid auction.