Appendix Z: Proof of Theorem 1

Set-up of the Optimal Program

For future reference, this appendix reproduces the optimization problem of the buyer with $U_{hH} = 0$ (Lemma 2) and with the subset of the IC constraints that happen to bind at the optimum.

$$\max_{\{x_k, q_k, U_k\}} \alpha_{lH} \left[x_{lH} W_{lH}(q_{lH}) - U_{lH}\right] + \alpha_{hH} x_{hH} W_{hH}(q_{hH}) + \alpha_{hL} \left[x_{hL} W_{hL}(q_{hL}) - U_{hL}\right] + \alpha_{lL} \left[x_{lL} W_{lL}(q_{lL}) - U_{lL}\right]$$

subject to:

$$\begin{aligned}
U_{lH} &\geq x_{hH} \Delta\theta_1 && \text{(IC 1)}\\
U_{hL} &\geq U_{lH} - x_{lH} \left[W_{lH}\left(q_{lH}\right) - W_{hL}\left(q_{lH}\right)\right] && \text{(IC 2)}\\
U_{hL} &\geq x_{hH} q_{hH} \Delta\theta_2 && \text{(IC 3)}\\
U_{lL} &\geq U_{lH} + x_{lH} q_{lH} \Delta\theta_2 && \text{(IC 4)}\\
U_{lL} &\geq U_{hL} + x_{hL} \Delta\theta_1 && \text{(IC 5)}\\
U_{lL} &\geq U_{hH} + x_{hH} \Delta\theta_1 + x_{hH} q_{hH} \Delta\theta_2 && \text{(IC 6)}\\
N \sum_{k \in K} \alpha_k x_k &\leq 1 - (1 - \sum_{k \in K} \alpha_k)^N \text{ for all subsets } K \text{ of } \{lH, hH, hL, lL\} && \text{(feasibility)}
\end{aligned}$$

(We omit the non exclusion constraint). The associated Lagrangian is given by:

$$\begin{aligned}
&\alpha_{lH} \left[x_{lH} W_{lH}(q_{lH}) - U_{lH}\right] + \alpha_{hH} x_{hH} W_{hH}(q_{hH}) + \alpha_{hL} \left[x_{hL} W_{hL}(q_{hL}) - U_{hL}\right] + \alpha_{lL} \left[x_{lL} W_{lL}(q_{lL}) - U_{lL}\right] \qquad (1)\\
&+\lambda_1 \left[U_{lH} - x_{hH} \Delta\theta_1\right] + \lambda_2 \left[U_{hL} - U_{lH} + x_{lH} \left(W_{lH}\left(q_{lH}\right) - W_{hL}\left(q_{lH}\right)\right)\right]\\
&+\lambda_3 \left[U_{hL} - x_{hH} q_{hH} \Delta\theta_2\right] + \lambda_4 \left[U_{lL} - U_{lH} - x_{lH} q_{lH} \Delta\theta_2\right] + \lambda_5 \left[U_{lL} - U_{hL} - x_{hL} \Delta\theta_1\right]\\
&\lambda_6 \left[U_{lL} - x_{hH} \Delta\theta_1 - x_{hH} q_{hH} \Delta\theta_2\right] - \sum \gamma_K \left[N \sum_{k \in K} \alpha_k x_k - 1 + (1 - \sum_{k \in K} \alpha_k)^N\right]
\end{aligned}$$

(where λ_i is the Lagrangian multiplier associated with IC constraint i, and γ_K is the multiplier associated with feasibility constraint K). Figure 13 provides a graphical representation of these IC constraints together with their associated multipliers. A dotted line means that a constraint may bind at the optimum. A full line means it always binds.

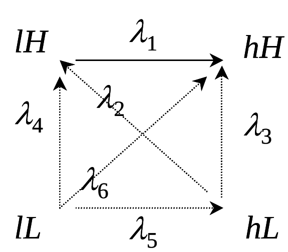

Figure 13: Potentially binding constraints at the solution

The Kuhn-Tucker conditions of this program are standard. For future reference, we only reproduce those with respect to U_k :

$$\lambda_1 - \lambda_2 - \lambda_4 = \alpha_{lH} \quad (2)$$
$$\lambda_2 + \lambda_3 - \lambda_5 = \alpha_{hL} \quad (3)$$
$$\lambda_4 + \lambda_5 + \lambda_6 = \alpha_{lL} \quad (4)$$

Characterization of the Optimal Buying Mechanism

Preliminaries

We first define the notation that we will be using for some of the x_k variables when they take specific values. When x_{lH} takes its maximum value conditional on lL keeping priority in the contract allocation, we will denote it $x_{lH}^{\max}$. Formally, $x_{lH}^{\max}$ is defined by the equation

$$N\left(\alpha_{lH}x_{lH}^{\max} + \alpha_{lL}x_{lL}^{FB}\right) = 1 - (\alpha_{hL} + \alpha_{hH})^N$$

By Border (1991), this implies the following allocation: When there is a type lL, give the contract to lL, if not, give priority to a type lH if there is one. Conversely, $x_{hL}^{\min}$ corresponds to the expected probability of winning for hL when lH and lL have priority over hL (but hL maintains priority over hH). Formally,

$$N\left(\alpha_{lH}x_{lH}^{\max} + \alpha_{hL}x_{hL}^{\min} + \alpha_{lL}x_{lL}^{FB}\right) = 1 - \alpha_{hH}^N$$

Finally, $\overline{x}$ is defined such that $x_{lH} = x_{hL}$ and they have priority over hH in the allocation, that is

$$N\left((\alpha_{lH} + \alpha_{hL})\overline{x} + \alpha_{lL}x_{lL}^{FB}\right) = 1 - \alpha_{hH}^N$$

The proof of Theorem 1 uses the following result repeatedly:

Lemma 7: *Suppose $U_{lH} = x_{hH}\Delta\theta_1$. (1) Suppose further that $U_{hL,lH} \geq U_{hL,hH}$. Then, $x_{hL} > x_{lH}$ if and only if $U_{lL,hL} > U_{lL,lH}$. (2) Suppose now that $U_{hL,lH} \leq U_{hL,hH}$. Then $U_{lL,hL} \geq U_{lL,lH}$ when $x_{hL} \geq x_{lH}$.*

Proof: The result follows directly from a comparison of $U_{lL,lH}$ and $U_{lL,hL}$ (when $U_{hL,lH} \geq U_{hL,hH}$) :

$$U_{lL,lH} = x_{lH}q_{lH}\Delta\theta_2 + x_{hH}\Delta\theta_1 \quad U_{lL,hL} = x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_1 + x_{lH}q_{lH}\Delta\theta_2 + x_{hH}\Delta\theta_1$$

When $U_{hL,hH} \geq U_{hL,lH}$, $U_{lL,hL} = x_{hL}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2$. Since $U_{hL,hH} \geq U_{hL,lH}$ is equivalent to $x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] \leq x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$, the condition implies $U_{lL,hL} \geq U_{lL,lH}$ when $x_{hL} > x_{lH}$. QED.

Lemma 8: *Suppose that $IC_{hL,hH}$ is satisfied. Then $x_{hL} \geq x_{hH} \Longrightarrow IC_{lL,hH}$ is satisfied.*

Proof: $\text{IC}_{hL,hH}$ satisfied means that $U_{lL,hL} \stackrel{\text{defn}}{=} U_{hL} + x_{hL}\Delta\theta_1 \geq U_{hH} + x_{hH}\Delta\theta_2 q_{hH} + x_{hL}\Delta\theta_1$. On this other hand, $U_{lL,hH} = U_{hH} + x_{hH}\Delta\theta_2 q_{hH} + x_{hH}\Delta\theta_1$. Clearly, $U_{lL,hH} \leq U_{lL,hL}$ as long as $x_{hL} \geq x_{hH}$. QED

We are now ready to prove theorem 1. The proof proceeds by progressively partioning the space of parameters into sets of parameters for which the solution shares the same binding IC and feasibility constraints. The logic of the proof is pretty simple, even if the mechanics can be involved. For this reason an exhaustive exposition of the proof of part I, scenario 1 is presented. The arguments in the rest of the proof are presented more briefly where they mirror those in part I, scenario 1.

Proof of part I of Theorem 1: $W_{lH}(\overline{q}) - W_{hL}(\overline{q}) > 0$ i.e. $\Delta\theta_1 > \overline{q}\Delta\theta_2$

The binding constraints in the buyer-optimal efficient mechanism are $\text{IC}_{lH,hH}$, $\text{IC}_{hL,hH}$ and $\text{IC}_{lL,hL}$. The buyer's resulting expected utility is given by

$$\begin{aligned}&\alpha_{lH}x_{lH}W_{lH}(q_{lH}) + \alpha_{hH}x_{hH}[W_{hH}(q_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}\Delta\theta_2] \\ &+\alpha_{hL}x_{hL}[W_{hL}(q_{hL}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] + \alpha_{lL}x_{lL}W_{lL}(q_{lL})\end{aligned} \tag{5}$$

(where, again, we have highlighted the virtual welfares associated with each type). Keeping the probabilities fixed at $x_k = x_k^{FB}$, optimizing the q's in (5) requires that only q_{hH} be adjusted away from the efficient level and set equal to

$$q_{hH}^2 = \arg\max\{W_{hH}(q_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}\Delta\theta_2\} \tag{6}$$

This reduces the informational rents of hL and lL. From Lemma 7(2), we know that $U_{lL,hL} \geq U_{lL,lH}$ as long as $U_{hL,hH} \geq U_{hL,lH}$. Hence, we need to consider only two scenarios:

Scenario 1: At q_{hH}^2, $U_{hL,hH} \geq U_{hL,lH}$, that is,

$$x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] \leq x_{lH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] \tag{7}$$

In this case, all IC constraints remain satisfied as we decrease q_{hH} to q_{hH}^2.
We now consider the optimization of the probabilities of winning. From (5) and the model assumptions, the virtual welfare associated with lL is the largest. Moreover, the virtual welfare associated with lH is larger than that associated with hH. Thus, we need to consider three cases depending on the relative ranking of the virtual welfare of hL with respect to the virtual welfares of hH and lH.

1. $VW_{hL} \geq VW_{lH} \geq VW_{hH} : W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{lH}(\overline{q}) > W_{hH}(q^2_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}} q^2_{hH}\Delta\theta_2$

   [Solution 1.1.a]

   The optimal probabilities of winning are $x_k = x_k^{FB}$ since the ranking of the virtual welfares corresponds to the ranking of the first best welfares. All IC constraints are satisfied given the arguments above. The x's and q's are optimized given the binding constraints; $q_{lH} = \overline{q}$, $q_{hH} = q^2_{hH}$ and $q_{hL} = q_{lL} = \underline{q}$.

2. $VW_{lH} > VW_{hL} \geq VW_{hH} : W_{lH}(\overline{q}) > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{hH}(q^2_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}} q^2_{hH}\Delta\theta_2$

   In this case, type lH generates a higher level of virtual welfare than type hL. Thus, the buyer would rather give the contract to supplier lH than to supplier hL, i.e. he would like to change the order of priority in the allocation. Increasing x_{lH} while decreasing x_{hL} concurrently (keeping $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL}^{FB}$ constant) does not initially affect any of the virtual welfare and it increases the buyer's expected utility. This process continues until either a new IC constraint binds or we have reach $x_{lH} = x_{lH}^{\max}$.

   We now argue that the only potentially new binding constraint is $\text{IC}_{lL,lH}$. To see this consider the following:

   (a) <u>hL's IC constraints:</u> Given that $U_{hL,lH} = U_{lH} - x_{lH}[\Delta\theta_1 - \Delta\theta_2\overline{q}]$ and that U_{lH} is not affected by the process, the incentives for hL to imitate lH have actually decreased. $\text{IC}_{hL,lL}$ remain satisfied as well since $\text{IC}_{lL,hL}$ is binding and $x_{lL} > x_{hL}$.

   (b) <u>lH's IC constraints:</u> Because $U_{lH,hL} = U_{hL} + x_{hL}(\Delta\theta_1 - \Delta\theta_2 q^2_{hH})$ and $U_{lH,lL} = U_{hL} + x_{hL}\Delta\theta_1 - x_{lL}\Delta\theta_2\underline{q}$, the incentives for lH to imitate hL and lL have decreased ($U_{hL} = x_{hH}\Delta\theta_1 q^2_{hH}$ is not affected by the process).

   (c) <u>hH's IC constraints:</u> hH continues to have no incentive to imitate hH, hL or lL given that $\text{IC}_{lH,hH}$ and $\text{IC}_{hL,hH}$ are binding, and $U_{hH,lL}$ is not affected by the process.

   (d) <u>lL's IC constraint:</u> By Lemma 8, $\text{IC}_{lL,hH}$ is not affected by the process. By Lemma 7(2), $\text{IC}_{lL,lH}$ remains satisfied as long as $x_{lH} \leq x_{hL}$, but it could start binding afterwards.

   Thus, we continue to increase x_{lH} at the cost of x_{hL} until either $x_{lH} = x_{lH}^{\max}$ or $\text{IC}_{lL,lH}$ starts binding, whichever comes first.

   (a) $x_{lH} = x_{lH}^{\max}$ first. **[Solution 1.1.b]**

   This means that $U_{lL,hL} \geq U_{lL,lH}$ even when x_{lH} reaches its maximum. This corresponds to the solution because there are no more opportunities to increase the buyer's expected utility: the q's are optimized given the binding IC constraints, the x's are optimized given the virtual welfare and the feasibility constraints. The solution is thus: $q_{lH} = \overline{q}$, $q_{hH} = q^2_{hH}$, $q_{hL} = q_{lL} = \underline{q}$ and $x_{lL} = x^{FB}_{lL} > x_{lH} = x^{\max}_{lH} > x_{hL} = x^{\min}_{hL} > x_{hH} = x^{FB}_{hH}$. By the argument just above, all IC constraints are satisfied.

(b) $\text{IC}_{lL,lH}$ starts binding. **[Solution 1.1.c]**

At that point, $U_{lL,lH} = U_{lL,hL}$, that is, $x^{FB}_{hH}[W_{lH}(q^2_{hH}) - W_{hL}(q^2_{hH})] = x_{hL}\Delta\theta_1 - x_{lH}\overline{q}\Delta\theta_2$ (note that by Lemma 7(2), this happens at $x_{lH} > x_{hL}$).

We now argue that we should be looking for a solution where both $\text{IC}_{lL,hL}$ and $\text{IC}_{lL,lH}$ are binding. Indeed, if only $\text{IC}_{lL,lH}$ binds, the virtual welfare associated with hL is W^{FB}_{hL} which is greater than the virtual welfare associated with lH. Thus the buyer would want to set x_{hL} back to x^{FB}_{hL}, but this would bring us back to the starting point.

Thus the buyer further increases his expected utility by increasing x_{lH} and decreasing x_{hL} while keeping $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x^{FB}_{lL}$ constant *and* $x^{FB}_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{hL}\Delta\theta_1 - x_{lH}\overline{q}\Delta\theta_2$. This requires that we adjust q_{hH} (specifically we need to increase q_{hH}).

A change in q_{hH} corresponds to a change in the value of the Lagrangian multiplier on the $\text{IC}_{lL,lH}$ constraint. Using the expressions in (1) to (4), we can rewrite the expressions for lH and hH's virtual welfares as follows:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{\lambda_4}{\alpha_{lH}}q_{lH}\Delta\theta_2\} \tag{8}$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\alpha_{lH}+\lambda_4}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_4}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \tag{9}$$

where λ_4 is the Lagrangian multiplier on the $\text{IC}_{lL,lH}$ constraint.

Thus, practically, we increase x_{lH} and decrease x_{hL} concurrently to keep $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x^{FB}_{lL}$ constant. This implies a new value for q_{hH} and q_{lH} to ensure that $x^{FB}_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{hL}\Delta\theta_1 - x_{lH}q_{lH}\Delta\theta_2$. These correspond to a new value for λ_4 through (9). Specifically, λ_4 increases.

This process increases the virtual welfare associated with hL, $W_{hL}(\overline{q}) - \frac{\alpha_{lL}-\lambda_4}{\alpha_{hL}}\Delta\theta_1$, and decreases the virtual welfare associated with lH and hH (see (8) and (9)).

It continues until we have either reached the upper bound to x_{lH}, $x^{\max}_{lH}$, or the virtual welfares associated with lH and hL become equal:

$$\max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{\lambda^*_4}{\alpha_{lH}}q_{lH}\Delta\theta_2\} = W_{hL}(\underline{q}) - \frac{\alpha_{lL}-\lambda^*_4}{\alpha_{hL}}\Delta\theta_1$$

whichever comes first. Thus $\lambda_4 \in (0, \lambda^*_4) \subset (0, \alpha_{lL})$ as required by (4).

This defines the solution: $x_{lL} = x^{FB}_{lL} > x^{\max}_{lH} \geq x_{lH} > x_{hL} \geq x^{\min}_{hL} > x_{hH} = x^{FB}_{hH}$, $q_{lL} = q_{hL} = \underline{q}$ and q_{lH} and q_{hH} defined by (8) and (9), q_{lH}, $q_{hH} < \overline{q}$. The x's are optimized given the virtual welfares and the feasibility constraints. The q's are optimized given the binding constraints.

All IC constraints remain satisfied. The arguments for this are the same as those we made above, except for $\text{IC}_{hL,lH}$, which follows because $x^{FB}_{hH}[W_{lH}(q_{hH})-W_{hL}(q_{hH})] \overset{U_{lL,hL}=U_{lL,lH}}{=} x_{hL}\Delta\theta_1 - x_{lH}q_{lH}\Delta\theta_2 < x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$ when $x_{lH} > x_{hL}$.

3. $VW_{lH} \geq VW_{hH} > VW_{hL} : W_{lH}(\overline{q}) > W_{hH}(q^2_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q^2_{hH}\Delta\theta_2 > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1$.

In this case, the ideal ordering of types in the allocation is $lL \succ lH \succ hH \succ hL$. The buyer increases his expected utility by decreasing x_{hL}, first to the benefit of x_{lH} (that is, keeping $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x^{FB}_{lL}$ constant), and then to the benefit of x_{hH} (that is, keeping $N(\alpha_{lH}x^{\max}_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x^{FB}_{lL} + \alpha_{hH}x_{hH}) = 1$).

This process initially does not affect any of the virtual welfares until a new IC constraint binds. By the same arguments as in point 2 above, we can establish that the first binding constraint is $\text{IC}_{lL,lH}$. When it binds $x_{hH}[\Delta\theta_1 - \Delta\theta_2 q^2_{hH}] = x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_2\overline{q}$. At this point, $x_{lH} > x_{hL} > x_{hH}$ (the first inequality comes from Lemma 7(2)).

Once this happens, any further improvement requires that we keep $U_{lL,hL} = U_{lL,lH}$ (otherwise, if $U_{lL,hL} < U_{lL,lH}$, $\text{IC}_{lL,hL}$ ceases to bind, the virtual welfare associated with hL bounces back to W^{FB}_{hL} and thus we get back to the starting point). We are thus in a similar situation as in point 2 above. Any further change in the x's requires some changes in the q's and thus in the value of the multiplier on the IC constraints. Using the expressions in (1) to (4), the resulting virtual welfares associated with lH, hH and hL are:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{\lambda_4}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} \tag{10}$$

$$VW_{hH} = \max_{hH}\{W_{hH}(q_{hH}) - \frac{(\alpha_{lH}+\lambda_4)}{\alpha_{hH}}\Delta\theta_1 - \frac{(\alpha_{hL}+\alpha_{lL}-\lambda_4)}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \tag{11}$$

$$VW_{hL} = W_{hL}(\underline{q}) - \frac{\alpha_{lL}-\lambda_4}{\alpha_{hL}}\Delta\theta_1 \tag{12}$$

where $\lambda_4 \in (0, \alpha_{lL})$ is such that $U_{lL,hL} = U_{lL,lH}$ i.e. $x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{hL}\Delta\theta_1 - x_{lH}q_{lH}\Delta\theta_2$ for the current value of x_{hL} (x_{lH} and x_{hH} are well-defined once x_{hL} is defined given that lH has priority hH is also clear). Practically, a decrease in x_{hL} is associated with an increase in q_{hH}, a decrease in q_{lH} and an increase in λ_4. This decreases VW_{lH} and VW_{hH} and increases VW_{hL}.

The difference relative to Solution 1.1.c is what ends this process. Here, the process ends when either a new IC constraint binds or the relative ranking of virtual welfare changes.[1] The only new IC constraint that can bind is $\text{IC}_{lL,hH}$. This happens at $x_{hL} = x_{hH}$. Thus we need to distinguish the following cases depending on which event happens first:

(a) We have reached $VW_{lH} \geq VW_{hH} = VW_{hL}$ and $x_{lH} = x_{lH}^{\max}$. Then this is the solution. The buyer is indifferent between hH and hL. The qualities are given by the value of λ_4 that solves for $VW_{hH} = VW_{hL}$ in (11) and (12), $q_{lL} = q_{hL} = \underline{q}$ and $x_{lL} = x_{lL}^{FB}$, $x_{lH} = x_{lH}^{\max} > x_{hL}^{\min} \geq x_{hL} > x_{hH} \geq x_{hH}^{FB}$. **[Solution 1.1.d]**

(b) We have reached $VW_{lH} \geq VW_{hH} = VW_{hL}$ at $x_{lH} < x_{lH}^{\max}$. Then the buyer can further increase his expected utility by decreasing x_{hL} and increasing x_{lH} keeping $U_{lL,lH} = U_{lL,hL}$. This further decreases VW_{lH} and VW_{hH} and increases VW_{hL}. The process stops when either $VW_{lH} = VW_{hL}$ or $x_{lH} = x_{lH}^{\max}$, whichever comes earlier. At the solution the q's are defined from (11) and (12) for the value of λ_4 at which the process stops, $q_{lL} = q_{hL} = \underline{q}$ and $x_{lL} = x_{lL}^{FB}$, $x_{lH}^{\max} \geq x_{lH} > x_{hL} \geq x_{hL}^{\min}$ and $x_{hH} = x_{hH}^{FB}$. This corresponds to **Solution 1.1.c.** above.

(c) We have reached $VW_{lH} = VW_{hH} > VW_{hL}$. (note that this implies that $q_{lH} < q_{hL}$ given (10) and (11)). The buyer further increases his expected utility by decreasing x_{hL} and adjusting x_{lH} and x_{hH} in a way that preserves $VW_{lH} = VW_{hH}$ and $U_{lL,lH} = U_{lL,hL}$.[2] Thus λ_4 is fixed and the virtual welfares are not affected. This process continues until $x_{hL} = x_{hH}$ $(< x_{lH})$ at which point $U_{lL,hH}$ starts binding. At this stage we have:

$$\begin{aligned} U_{lL,lH} &= x_{hH}\Delta\theta_1 + x_{lH}q_{lH}\Delta\theta_2 = U_{lL,hH} = x_{hH}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2 \\ &= U_{lL,hL} = x_{hL}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2 \end{aligned}$$

Using the expressions in (1) to (4), the virtual welfares are given by

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{\lambda_4}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} \tag{13}$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH} - \frac{\alpha_{lH} + \lambda_4}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL} + \alpha_{lL} - \lambda_4}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \tag{14}$$

$$VW_{hL} = W_{hL}(\underline{q}) - \frac{\alpha_{lL} - \lambda_4 - \lambda_6}{\alpha_{hL}}\Delta\theta_1 \tag{15}$$

where λ_4 and λ_6 are the multipliers on the $\text{IC}_{lL,lH}$ and $\text{IC}_{lL,hH}$ constraint respectively.

[1] No feasibility constraint binds in the process. Indeed, the only potential feasibility constraint would involve x_{hH} hitting its maximum but this never occurs before $x_{hH} = x_{hL}$.

[2] The feasibility constraints on the x's are $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hH}x_{hH}) \leq 1 - \alpha_{hL}^N$ and $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$

There exists a value for λ_4 and λ_6 such that $VW_{lH} = VW_{hH} = VW_{hL}$ and $U_{lL,lH} = U_{lL,hL} = U_{lL,hH}$ and $N(\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL}^{FB} + \alpha_{hH}x_{hH}) = 1$. Indeed, we have five equations and five unknowns: λ_4, λ_6, x_{lH}, x_{hL} and x_{hH} (from (15) and the fact $VW_{lH} = VW_{hL}$, we know that $\alpha_{lL} - \lambda_4 - \lambda_6 > 0$, thus $\alpha_{hL} + \alpha_{lL} - \lambda_4$ in (14) is ensured to be positive which is required by the non negative constraint on the multipliers).

These values for λ_4 and λ_6 correspond to the solution. At the solution, $x_{lH} > x_{hH} = x_{hL}$ (implied by $U_{lL,lH} = U_{lL,hL} = U_{lL,hH}$), $q_{lH} < q_{hL} < \overline{q}$ and $q_{hL} = q_{hH} = \underline{q}$. The buyer is indifferent among lH, hH and hL and the x's are thus optimized. The q's are optimized given the binding constraints and the value of the multipliers. No new constraint binds in the process. The argument for this is identical as the one in point 2, except for $\text{IC}_{hL,lH}$, which follows because $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] \overset{U_{lL,hL}=U_{lL,lH}}{=} x_{hL}\Delta\theta_1 - x_{lH}q_{lH}\Delta\theta_2 < x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$ when $x_{lH} > x_{hL}$. **[Solution 1.1.e]**

(d) We have reached $x_{hH} = x_{hL}$. At this point, $\text{IC}_{lL,hH}$ starts binding. The rest of the argument is as in point c above: There exists a value for λ_4 and λ_6 such that $VW_{lH} = VW_{hH} = VW_{hL}$ and $U_{lL,lH} = U_{lL,hL} = U_{lL,hH}$ and $N(\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL}^{FB} + \alpha_{hH}x_{hH}) = 1$. The solution is thus Solution 1.1.e.

Scenario 2: At q_{hH}^2, $U_{hL,hH} < U_{hL,lH}$, that is, $x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] > x_{lH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$.

In this case, $\text{IC}_{hL,lH}$ becomes binding as we decrease q_{hH}. To reduce hL and lL's rents further, one now needs to decrease q_{lH} at the same time as q_{hH} in such a way that $U_{hL,hH} = U_{hL,lH}$, i.e., $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{lH}^{FB}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$. (Note that this implies that $q_{lH} > q_{hH}$.) Formally, using (1) to (4) in Appendix A, we let q_{lH} and q_{hH} solve:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) + \frac{\lambda_2^*}{\alpha_{lH}}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]\} \tag{16}$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\alpha_{lH} + \lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{(\alpha_{hL} + \alpha_{lL} - \lambda_2^*)}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \tag{17}$$

for the value of $\lambda_2^* \in (0, \alpha_{hL} + \alpha_{lL})$ such that $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{lH}^{FB}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$ (λ_2 is the multiplier on $\text{IC}_{hL,lH}$). Such value for λ_2 always exists. When $\lambda_2^* = 0$, $q_{lH} = \overline{q}$ and $q_{hH} = q_{hH}^2$ so that $x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] > x_{lH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$ from the definition of scenario 2. When $\lambda_2^* = \alpha_{hL} + \alpha_{lL}$, $q_{lH} < q_{hH} = \overline{q}$ and $x_{hH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] < x_{lH}^{FB}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)]$).

Relative to the BOEM, only the rents of hL and lL have decreased. The IC constraint of hL is taken care of by construction, and $U_{lL,hL} \geq U_{lL,lH}$ from Lemma 7(1). Hence, all IC constraints remain satisfied.

We now optimize over the x's. Notice that $VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH})+\frac{\lambda_2^*}{\alpha_{lH}}[W_{lH}(q_{lH})-W_{hL}(q_{lH})]\} > W_{lH}(\overline{q}) > VW_{hH}$. Hence, we need to consider three cases depending on the relative ranking of the virtual welfare associated with hL.

1. $VW_{hL} \geq VW_{lH} > VW_{hH}$: $W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq \max_{q_{lH}}\{W_{lH}(q_{lH}) + \frac{\lambda_2^*}{\alpha_{lH}}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]\}$ **[Solution 1.2.a]**

   The optimal probabilities are thus $x_k = x_k^{FB}$. The values of q_{lH} and q_{hH} are defined in (16) and (17) and $\overline{q} > q_{lH} > q_{hH} > q_{hH}^2$, $q_{hL} = q_{lL} = \underline{q}$.

2. $VW_{lH} > VW_{hL} \geq VW_{hH}$: $\max_{q_{lH}}\{W_{lH}(q_{lH}) + \frac{\lambda_2^*}{\alpha_{lH}}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]\} > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{hH}(q_{hH}^2) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}^2\Delta\theta_2$ (note that the condition is on VW_{hH} evaluated at $\lambda_2 = 0$).

   At the current value of λ_2, the buyer prefers to give the contract to lH over hL. As we progressively increase x_{lH} at the expense of x_{hL}, while keeping $x_{hH}^{FB}[W_{lH}(q_{hH})-W_{hL}(q_{hH})] = x_{lH}[W_{lH}(q_{hL}) - W_{hL}(q_{hL})]$, we decrease λ_2 (i.e. increase q_{lH} and decrease q_{hH} - from (16) and (17)). This decreases VW_{lH} and increases VW_{hH}.

   This process continues until the relative ordering of virtual welfares changes or the binding IC constraints change (at least of one these two events happen before we reach the feasibility constraint $x_{lH} = x_{lH}^{\max}$). Specifically, the two IC constraints we need to worry about are $\text{IC}_{hL,lH}$ which stops binding when $\lambda_2 = 0$, and $\text{IC}_{lL,lH}$ which starts binding when $x_{lH} = x_{hL}$. This yields three cases depending on which event happens first:

   (a) $VW_{lH} = VW_{hL}$ first (note that given the assumption of this case, $VW_{hL} \geq VW_{hH}$ always): We have then reached the solution. At the solution, the probabilities of winning are: $x_{lL} = x_{lL}^{FB} > x_{hL}^{FB} > x_{hL} > x_{lH} > x_{lH}^{FB} > x_{hH} = x_{hH}^{FB}$ where x_{lH} and x_{hL} are defined implicitly by $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$ for the values of q_{hH} and q_{lH} that solve (16) and (17) at the current value of λ_2 ($q_{hH} < q_{lH}$). The x's are optimized given the virtual welfares. The q's are optimized given the binding constraints and the value of λ_2. **[Solution 1.2.b]**

   (b) $\lambda_2 = 0$ first. $\text{IC}_{hL,lH}$ ceases to bind and $q_{hH} = q_{hH}^2$ and $q_{lH} = \overline{q}$. As x_{lH} further increases and x_{hL} decreases, the buyer increases his expected utility. None of the virtual welfares are affected in the process, and thus this continues until we either reach $x_{lH} = x_{lH}^{\max}$ or $\text{IC}_{lL,lH}$ starts binding (this happens when $x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] = x_{hL}\Delta\theta_1 - x_{hL}\Delta\theta_2\overline{q}$.).

In the first case, we are as in **Solution 1.1.b:** $x_{lL} = x_{lL}^{FB} > x_{lH} = x_{lH}^{\max} > x_{hL} = x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$, $q_{hH} = q_{hH}^2$ and $q_{lH} = \overline{q}$. The x's are optimized given that, by assumption, $VW_{hL} \geq VW_{hH}$.

In the second case, we are as in **Solution 1.1c.** Thus, $x_{lL} = x_{lL}^{FB} > x_{lH}^{\max} \geq x_{lH} > x_{hL} \geq x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$, $q_{lL} = q_{hL} = \underline{q}$ and q_{lH} and q_{hH} defined by (8) and (9), q_{lH}, $q_{hH} < \overline{q}$.

(c) $x_{lH} = x_{hL}$ first. At this point, $\text{IC}_{lL,lH}$ starts binding. Based on the expressions from (1), reworked using the equalities (2) to (4), the associated virtual welfares are given by:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{(\alpha_{hL} + \alpha_{lL} - \lambda_3)}{\alpha_{lH}}\Delta\theta_2 q_{lH} + \frac{\alpha_{hL} + \lambda_5 - \lambda_3}{\alpha_{lH}}\Delta\theta_1\} \quad (18)$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\lambda_3}{\alpha_{hH}}\Delta\theta_2 q_{hH} - \frac{\alpha_{lH} + \alpha_{hL} + \alpha_{lL} - \lambda_3}{\alpha_{hH}}\Delta\theta_1\} \quad (19)$$

$$VW_{hL} = W_{hL}(\underline{q}) - \frac{\lambda_5}{\alpha_{hL}}\Delta\theta_1 \quad (20)$$

There exist values for λ_3 and λ_5 such that (1) $\overline{x}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$ and (2) $VW_{lH} = VW_{hL}$. To see this, note that the progressive adjustment of x_{lH} until $x_{lH} = x_{hL}$ implies that there exists a value for λ_3 that satisfies condition (1). Once λ_3 is fixed, there is a value of λ_5 that ensures condition (2). Indeed for any feasible λ_3, when $\lambda_5 = 0$, the virtual welfare of hL is greater. When $\lambda_5 = \alpha_{lL}$ and $\lambda_2 = \alpha_{hL} + \alpha_{lL} - \lambda_3$, this follows from the fact that we have assume that $VW_{lH} > VW_{hL}$ when $\text{IC}_{lL,lH}$ becomes binding.

Note that $\lambda_2 = \alpha_{hL} - \lambda_3 + \lambda_5$. If the implied λ_2 is positive, this is the solution: $x_{lL} = x_{lL}^{FB} > x_{hL}^{FB} > x_{hL} = \overline{x} = x_{lH} > x_{lH}^{FB} > x_{hH} = x_{hH}^{FB}$ and the q's solving (18) through (20) above for the values of λ_3 and λ_5 that satisfy conditions (1) and (2) (in particular, $q_{lH} > q_{hH}$). The x's are optimized given the virtual welfares: the buyer is indifferent between lH and hL and $VW_{lH} > VW_{hH}$ follows from the comparison between (18) and (19) when $q_{lH} > q_{hH}$. The q's are optimized given the binding constraints and the value of the multipliers. [**Solution 1.2.c**]

If the implied λ_2 is strictly negative, then $\text{IC}_{hL,lH}$ ceases to bind at some point. We are then in the same situation as in **Solution 1.1.c**. At the solution, $x_{lL} = x_{lL}^{FB} > x_{lH}^{\max} \geq x_{lH} > x_{hL} \geq x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$, $q_{lL} = q_{hL} = \underline{q}$ and q_{lH} and q_{hH} defined by (8) and (9), q_{lH}, $q_{hH} < \overline{q}$.

3. $VW_{lH} > VW_{hH} > VW_{hL} : W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 < W_{hH}(q_{hH}^2) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}^2\Delta\theta_2$ (note that the condition is on VW_{hH} evaluated at $\lambda_2 = 0$).

In this case, we ideally want to decrease x_{hL}, first to the benefit of x_{lH} (then, possibly to the benefit of x_{hH}). Doing this while keeping $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$, requires that we decrease λ_2 (cf. (16) and (17)). This decreases VW_{lH} and increases VW_{hH}, but given the condition on this case, the ordering of virtual welfares is not affected. Thus, this process continues until, either we reach $\lambda_2 = 0$ (and thus IC$_{hL,lH}$ ceases to bind) or $x_{lH} = x_{hL}$ (and thus IC$_{lL,lH}$ starts binding).

(a) We reach $x_{lH} = x_{hL}$ when $\lambda_2 > 0$: This implies that $IC_{lL,lH}$ becomes binding in the process. Optimizing from now on with constraints $IC_{lH,hH}$, $IC_{lL,lH}$, $IC_{lL,hL}$, $IC_{hL,lH}$ and $IC_{hL,hH}$ binding requires that we keep $x_{lH} = x_{hL}$. The virtual welfares are given by (18), (19) and (20). Like in part 1, scenario 2, case 2c, we proceed by first looking for values of λ_3, λ_5 and q's such that (1) $\overline{x}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$, i.e. $U_{lL,hL} = U_{lL,lH}$ and $U_{hL,hH} = U_{hL,hH}$ and (2) $VW_{lH} = VW_{hL}$.

   If the implied λ_2 is positive, then this is the solution (solution 1.2.c) because condition (1) implies that $q_{lH} > q_{hH}$, which in turn ensures that $VW_{lH} = VW_{hL} > VW_{hH}$. The x's are optimized, and so are the q's.

   If the implied λ_2 is negative, then we are as in part I, scenario 1, case 3: the binding constraints are IC$_{lL,lH}$, IC$_{lL,hL}$, IC$_{lH,hH}$ and IC$_{hL,hH}$. This leads to solutions 1.1.c, 1.1.d or 1.1.e.

(b) We reach $\lambda_2 = 0$ when $x_{lH} \leq x_{hL}$. We can continue to increase x_{lH} at the expense of x_{hL}, and afterwards if necessary increase x_{hH} at the expense of x_{hL} until $IC_{lL,lH}$ starts binding. ($IC_{hL,lH}$ no longer binds because increasing x_{lH} beyond x_{hL} means that $x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] < x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$). The case then reduces to part 1, scenario 1, case 3, implying one of solutions 1.1.c, 1.1.d or 1.1.e apply.

Proof of part II of Theorem 1: $W_{lH}(\overline{q}) - W_{hL}(\overline{q}) < 0$ **i.e.** $\Delta\theta_1 < \overline{q}\Delta\theta_2$

The binding constraints in the buyer-optimal efficient mechanism are IC$_{lH,hH}$, IC$_{hL,lH}$ and IC$_{lL,hL}$. The buyer's resulting expected utility is given by

$$\alpha_{lH}x_{lH}[W_{lH}(q_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}\Delta\theta_2] + \alpha_{hH}x_{hH}[W_{hH}(q_{hH}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1]$$
$$+\alpha_{hL}x_{hL}[W_{hL}(q_{hL}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] + \alpha_{lL}x_{lL}W_{lL}(q_{lL}) \tag{21}$$

Keeping the probabilities fixed at $x_k = x_k^{FB}$, optimizing the q's requires that q_{lH} be set equal to

$$q_{lH}^2 = \arg\max\{W_{lH}(q_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}\Delta\theta_2\} \tag{22}$$

This reduces the informational rents of hL and lL. By Lemma 7(1), we know that $U_{lL,hL} > U_{lL,lH}$ as long as $U_{hL,lH} \geq U_{hL,hH}$. Hence, we need to consider only two scenarios, depending on whether $IC_{hL,hH}$ binds at q^2_{LH}:

Scenario 1: At q^2_{lH}, $U_{hL,lH} \geq U_{hL,hH}$, i.e., $x^{FB}_{lH}[W_{lH}(q^2_{lH}) - W_{hL}(q^2_{lH})] \leq x^{FB}_{hH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$ In this case, all IC constraints remain satisfied as we decrease q_{lH} to q^2_{lH}. Note that $W_{lH}(q^2_{lH}) - W_{hL}(q^2_{lH}) \equiv \Delta\theta_1 - \Delta\theta_2 q^2_{lH} < 0$. We now consider the optimization of the probabilities of winning. From (21), the virtual welfare associated with lL is the largest. This leaves four cases depending on the relative ranking of lH, hH and hL:

1. $VW_{hL} \geq VW_{lH} \geq VW_{hH} : [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] \geq [W_{lH}(q^2_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}} q^2_{lH}\Delta\theta_2] \geq [W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1]$ **[Solution 2.1.a]**

   The optimal probabilities of winning are $x_k = x^{FB}_k$ since the ranking of the virtual welfares corresponds to the ranking of the first best welfares. All IC constraints are satisfied. The x's and q's are optimized given the binding constraints.

2. $VW_{lH} > VW_{hH} \geq VW_{hL} : [W_{lH}(q^2_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}} q^2_{lH}\Delta\theta_2] > W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1 \geq [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1]$; or
   $VW_{lH} > VW_{hL} \geq VW_{hH} : [W_{lH}(q^2_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}} q^2_{lH}\Delta\theta_2] > [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] \geq W_{hH}(q_{hH}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1$

   The buyer would like to increase x_{lH} at the expense of x_{hL}. Doing this does not affect the supplier hL's IC constraint: $U_{hL,lH} \geq U_{hL,hH}$ corresponds to $x_{lH}[W_{lH}(q^2_{lH}) - W_{hL}(q^2_{lH})] \leq x^{FB}_{hH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$ and $W_{lH}(q^2_{lH}) - W_{hL}(q^2_{lH}) < 0$. Moreover, as long as $x_{hL} > x_{lH}$, the change in x_{lH} does not affect lL's IC constraint either (Lemma 7(1)). Thus, changing x_{lH} does not initially affect the virtual welfares.

   When we reach $x_{lH} = x_{hL} = \overline{x}$, $\text{IC}_{lL,lH}$ starts binding since $U_{lL,hL} = x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_1 + x_{lH}q^2_{lH}\Delta\theta_2 + x_{hH}\Delta\theta_1$ and $U_{lL,lH} = x_{lH}q^2_{lH}\Delta\theta_2 + x_{hH}\Delta\theta_1$. Define $\lambda^*_5 \in (0, \alpha_{lL})$, the value of λ_5 that equalizes the virtual welfares associated with lH and hL:

   $$W_{lH}(q^2_{lH}) + \frac{(\alpha_{hL} + \lambda^*_5)}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL} + \alpha_{lL}}{\alpha_{lH}}\Delta\theta_2 q^2_{lH} = W_{hL}(\underline{q}) - \frac{\lambda^*_5}{\alpha_{hL}}\Delta\theta_1 \tag{23}$$

   (from (1) to (4)). Such a value for λ_5 exists. When $\lambda_5 = 0$, the virtual welfare associated with hL is larger. When $\lambda_5 = \alpha_{lL}$, the virtual welfare of lH is bigger by assumption. Note that this process does not affect the virtual welfare associated with hH, which remains unchanged.

   (a) **[Solution 2.1.b]** If at λ^*_5, $VW_{lH} = VW_{hL} > VW_{hH}$, then the solution is $q_{lH} = q^2_{lH}$, $q_{hH} = \overline{q}$ and $q_{hL} = q_{lL} = \underline{q}$ and $x_{lL} = x^{FB}_{lL}, x_{hH} = x^{FB}_{hH}$, and $x_{lH} = x_{hL} = \overline{x}$. All IC constraints are satisfied. The q's and the x's are optimized given the binding constraints (in particular, the buyer is indifferent between lH and hL, but strictly prefer these to hH).

(b) If at λ_5^*, $VW_{lH} = VW_{hL} < VW_{hH}$, the buyer prefers hH to lH or hL. He increases his expected utility by raising x_{hH} while keeping $U_{lL,lH} = U_{lL,hL}$, that is, $x_{lH} = x_{hL}$, and $\lambda_5 = \lambda_5^*$. This process does not initially affect any of the virtual welfares until $\text{IC}_{hL,hH}$ starts binding (this happens at $x_{hL} = x_{lH} > x_{hH}$ given that $q_{lH} = q_{lH}^2 < q_{hH} = \overline{q}$ when $U_{hL,hH} \leq U_{hL,lH}$).

From then on, $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding. The expressions for the resulting virtual welfares are given by:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{(\alpha_{hL} + \alpha_{lL} - \lambda_3)}{\alpha_{lH}}\Delta\theta_2 q_{lH} + \frac{\alpha_{hL} + \lambda_5 - \lambda_3}{\alpha_{lH}}\Delta\theta_1\} \quad (24)$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\lambda_3}{\alpha_{hH}}\Delta\theta_2 q_{hH} - \frac{\alpha_{lH} + \alpha_{hL} + \alpha_{lL} - \lambda_3}{\alpha_{hH}}\Delta\theta_1\} \quad (25)$$

$$VW_{hL} = W_{hL}(\underline{q}) - \frac{\lambda_5}{\alpha_{hL}}\Delta\theta_1 \quad (26)$$

The buyer increases his expected utility by continuing to increase x_{hH} at the cost of x_{hL} and x_{lH}, while satisfying: (1) $U_{lL,lH} = U_{lL,hL}$ (thus $x_{lH} = x_{hL}$), (2) $U_{hL,hH} = U_{hL,lH}$, that is $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$, and (3) $VW_{lH} = VW_{hL}$. This requires an increase in λ_3 and a decrease in λ_5, i.e. a rise in q_{lH} and a decrease in q_{hH} (nonetheless, $q_{lH}^2 < q_{lH} < q_{hH}$ remains as long as $VW_{lH} \leq VW_{hH}$ as is apparent from (24) and (25)).[3]

This process stops when either $VW_{hH} = VW_{lH} = VW_{hL}$ or we hit a non negativity constraint for the multiplier $\lambda_2 = \alpha_{hL} + \lambda_5 - \lambda_3$.

i. **[Solution 2.1.d]** Suppose $VW_{hH} = VW_{lH} = VW_{hL}$ at a point where $\lambda_2 \geq 0$. Then we have reached the solution. The q's are defined from (24) and (25) for the values of λ_3 and λ_5 that equalize the virtual welfares (note that this implies that $q_{lH} < q_{hH}$, so that, in turn, $U_{hL,hH} = U_{hL,lH}$ implies $x_{lH} > x_{hH}$ as required for incentive compatibility). The x's are such that $x_{lL} = x_{lL}^{FB}$, and $x_{lH}^{FB} > x_{lH} = x_{hL} > x_{hH} > x_{hH}^{FB}$ with $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$.[4] All IC constraints are satisfied. The q's are optimized given the binding constraints. The

[3] Formally, we have four equations (the three constraints mentioned in the text, plus the feasibility constraint $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$) and five unknowns: x_{hH}, x_{hL}, x_{lH} and λ_3 and λ_5 (the q's are determined on the basis of the λ's by (24) and (25)). Thus any value for x_{hH} pins down the other variables.

[4] No other feasibility constraint for the probabilities of winning binds, except for the one-type constraint for x_{lL}.

x's are optimized given the resulting virtual welfares (the buyer is indifferent among lH, hL and hH).

ii. **[Solution 2.1.e]** Suppose λ_2 reaches zero at a point where $VW_{hH} > VW_{lH} = VW_{hL}$.

Let λ_5^{**}, the value of λ_5 at this point. We also have $q_{lH}^2 < q_{lH} < q_{hH}$ and $x_{lH} = x_{hL} > x_{hH}$ at this point. The buyer further increases his utility by increasing x_{hH} at the cost of x_{lH} and x_{hL}, while keeping $U_{lL,lH} = U_{lL,hL}$ and $VW_{lH} = VW_{hL}$ (i.e. $\lambda_5 = \lambda_5^{**}$ and the q's are fixed at $q_{lH} < q_{hH}$).[5] This process at first does not affect the virtual welfares (since λ_5 is fixed, we keep having $VW_{hH} > VW_{lH} = VW_{hL}$), until IC$_{lL,hH}$ starts binding.[6] At this stage we have:

$$\begin{aligned} U_{lL,lH} &= x_{hH}\Delta\theta_1 + x_{lH}q_{lH}\Delta\theta_2 = U_{lL,hH} = x_{hH}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2 \\ &= U_{lL,hL} = x_{hL}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2 \end{aligned}$$

thus $x_{hL} = x_{hH} < x_{lH}$. To keep increasing the buyer's welfare while satisfying all three constraints out of lL requires that we keep $x_{hL} = x_{hH}$. Thus we increase both x_{hL} and x_{hH} at the expense of x_{lH} (this will indeed increase the buyer's utility since $VW_{hH} > VW_{lH} = VW_{hL}$), and adjust the q's as needed, that is, we increase q_{lH} and decrease q_{hH}. We do this until $VW_{lH} = VW_{hL} = VW_{hH}$. We have then reached the solution. At the solution, $q_{lH} < q_{hH}$ and $x_{lL} = x_{lL}^{FB}$, and $x_{lH}^{FB} > x_{lH} > x_{hL} = x_{hH} > x_{hH}^{FB}$ with $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$.

3. $VW_{hL} > VW_{hH} > VW_{lH} : [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] > [W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1] > [W_{lH}(q_{lH}^2) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}^2\Delta\theta_2]$; or
$VW_{hH} > VW_{hL} > VW_{lH} : W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1 > [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] > [W_{lH}(q_{lH}^2) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}^2\Delta\theta_2]$

In this case, the buyer would like to increase x_{hH} at the expense of x_{lH}. As we increase x_{hH} and decrease x_{lH}, we reach a point where $x_{lH}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)] = x_{hH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$, that is, IC$_{hL,hH}$ starts binding.

A candidate solution is defined by the value of $\lambda_2 \in (0,\, \alpha_{hL} + \alpha_{lL})$ that equates VW_{lH} and

[5] The exact way in which x_{lH} and x_{hL} are decreased is determined by $U_{lL,lH} = U_{lL,hL}$, i.e. $x_{lH}q_{lH}\Delta\theta_2 + x_{hH}\Delta\theta_1 = x_{hL}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2$ and the feasibility constraint $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$.

[6] This is the only constraint that can bind in the process. No new constraint can bind out of lH since $U_{lH} = x_{hH}\Delta\theta_1$ increases and alternatives decrease. No new constraint can bind out of hL because $\Delta\theta_1 - \Delta\theta_2 q_{hH} < 0$ given that $q_{hH} > q_{lH} > q_{lH}^2$ and $VW_{hH} \geq VW_{lH}$.

VW_{hH} :

$$\max_{q_{lH}}\{W_{lH}(q_{lH})+\frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1-\frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} = \max_{q_{hH}}\{W_{hH}(q_{hH})-\frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1-\frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \quad (27)$$

(from (1) to (4)). Such value for λ_2 exists since the virtual welfare of lH is larger than that of hH at $\lambda_2 = 0$, and smaller at $\lambda_2 = \alpha_{hL} + \alpha_{lL}$ by assumption. By inspection of (27), this happens at $\frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}} < \frac{\lambda_2^*}{\alpha_{lH}}$ that is, the resulting q's are such that $q_{lH}^2 < q_{lH} < q_{hH}$. Finally, we require that $U_{hL,lH} = U_{hL,hH}$, that is, $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$ which implies that $x_{lH} > x_{hH}$ as required by incentive compatibility.

This process only affected VW_{hL} and VW_{hH}. If $VW_{hL} > VW_{hH} = VW_{lH}$ at this point, then this is indeed the solution. The other variables are set such that $x_{lL} = x_{lL}^{FB}$, $x_{hL} = x_{hL}^{FB}$, and $q_{hL} = q_{lL} = \overline{q}$. The q's are optimized given the values of the multipliers and the binding constraints. The x's are optimized given the resulting virtual welfares. All IC constraints are satisfied ($\text{IC}_{lL,lH}$ satisfied given Lemma 7(1)). **[Solution 2.1.c]**

If $VW_{hL} < VW_{hH} = VW_{lH}$, the buyer can further increase his expected utility by increasing x_{hH} and x_{lH} at the cost of x_{hL}. He does so while keeping $\lambda_2 = \lambda_2^*$ so that $VW_{hH} = VW_{lH}$. The exact way in which x_{hH} and x_{lH} are increased is pinned down by $x_{lH}[W_{lH}(q_{lH})-W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$. This process does not affect the virtual welfare, until $x_{hL} = x_{lH}$ at which point $\text{IC}_{lL,lH}$ starts binding. We are now in a situation where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding and $VW_{hL} < VW_{hH} = VW_{lH}$. From then on, the virtual welfares are those defined in (24) - (26). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in 2(b) above, and the proof thus proceeds along the same lines: we look for a solution where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are binding and $VW_{hL} = VW_{hH} = VW_{lH}$, or $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{lL,hH}$ and $\text{IC}_{hL,hH}$ are binding and $VW_{hL} = VW_{hH} = VW_{lH}$. **[Solution 2.1.d or 2.1.e]**

4. $VW_{hH} > VW_{lH} > VW_{hL}$: $W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1 > [W_{lH}(q_{lH}^2) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}} q_{lH}^2\Delta\theta_2] > [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1]$

   Given the ordering of virtual welfares, the buyer is first tempted to increase x_{hH} at the expense of x_{hL}.[7] Two things can happen in the process: (1) $\text{IC}_{lL,lH}$ starts binding (this happens at $x_{lH}^{FB} = x_{hL}$ because $U_{lL,lH} = x_{lH}\Delta\theta_2 q_{lH}^2 + x_{hH}\Delta\theta_1$ and $U_{lL,hL} = x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_1 + x_{lH}\Delta\theta_2 q_{lH}^2 + x_{hH}\Delta\theta_1$), (2) $\text{IC}_{hL,hH}$ starts binding (this happens at a point where

[7] That is, keeping the equality $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH}^{FB} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$.

$x_{hH} < x_{lH}^{FB}$ since $x_{hH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] = x_{lH}^{FB}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)]$ at that point, and $W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2) < 0$ from the definition of scenario 1). We examine each case in turn.

(a) $IC_{lL,lH}$ binds first ($x_{lH}^{FB} = x_{hL}$)

Let λ_5^*, the value of λ_5 that equalizes VW_{lH} and VW_{hL}. This was defined in (23). We now have $VW_{hH} > VW_{lH} = VW_{hL}$. Thus the buyer can increase his welfare by increasing x_{hH}. The rest of the solution is as described in 2(b) above. [**Solution 2.1.d or Solution 2.1.e**].

(b) $\text{IC}_{hL,hH}$ binds first:

This happens at $x_{hL} > x_{lH}^{FB} > x_{hH}$ (the first inequality comes from the fact that $\text{IC}_{hL,hH}$ binds first; the second inequality comes from the fact that $q_{lH} < q_{hH} = \overline{q}$ at the point where $\text{IC}_{hL,hH}$ starts binding). Increasing further x_{hH} at the expense of x_{hL}, while keeping $x_{lH}^{FB}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$ requires that we decrease q_{hH} and increase q_{lH}. This corresponds to a rise in λ_3, a decrease in VW_{hH} and an increase in VW_{lH}. This process stops when either $VW_{lH} = VW_{hH}$ or $x_{lH} = x_{hL}$ whichever comes first (note at this stage $x_{lH} = x_{hL} > x_{hH}$ and $IC_{lL,lH}$ starts binding). If $VW_{lH} = VW_{hH}$ first, we can continue to increase the buyer's utility by decreasing x_{hL}, this time to the benefit of both lH and hH while keeping $VW_{lH} = VW_{hH}$ and $U_{hL,hH} = U_{hL,lH}$ (note that this implies $q_{lH} < q_{hH}$ and $x_{hL} > x_{hH}$). This process continues until $x_{hL} = x_{lH}$ at which point $IC_{lL,lH}$ starts binding.

Thus, in both events, we reach a point where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding. From then on, the virtual welfares are those defined in (24) - (26). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in 2(b) above, and the proof thus proceeds along the same lines: we look for a solution where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are binding and $VW_{hL} = VW_{hH} = VW_{lH}$, or $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{lL,hH}$ and $\text{IC}_{hL,hH}$ are binding and $VW_{hL} = VW_{hH} = VW_{lH}$. [**Solution 2.1.d or 2.1.e**]

Scenario 2: At q_{lH}^2, $U_{hL,hH} > U_{hL,lH}$ that is, $x_{lH}^{FB}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)] > x_{hH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$ In this case, $\text{IC}_{hL,hH}$ becomes binding as we decrease q_{lH} towards q_{lH}^2. To decrease the rents of hL and lL, we now need to decrease q_{lH} and q_{hH}, holding $U_{hL,hH} = U_{hL,lH}$. The optimal q's are defined by:

$$
\begin{aligned}
q_{lH}^{*} &= \arg\max_{q_{lH}}\{W_{lH}(q_{lH}) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} \\
q_{hH}^{*} &= \arg\max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}\}
\end{aligned}
$$

where $\lambda_2^* \in (0, \alpha_{hL} + \alpha_{lL})$ is chosen such that $x_{lH}^{FB}[W_{lH}(q_{lH}^*) - W_{hL}(q_{lH}^*)] = x_{hH}^{FB}[W_{lH}(q_{hH}^*) - W_{hL}(q_{hH}^*)]$. Note that the sign of $W_{lH}(q_{lH}^*) - W_{hL}(q_{lH}^*) = \Delta\theta_1 - \Delta\theta_2 q_{lH}^*$ is not pinned down *a priori* so that q_{lH} and q_{hH} cannot be ranked. No other new constraint binds in the process (Lemma 7(1)).

We now consider the optimization of the probabilities of winning. We need to consider five cases:

1. $VW_{hL} \geq VW_{lH} \geq VW_{hH}$: $W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{lH}(q_{lH}^*) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}^* \geq W_{hH}(q_{hH}^*) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}^*$.

   The optimal probabilities of winning are $x_k = x_k^{FB}$. This corresponds to **Solution 1.2.a** except that q_{lH} and q_{hH} cannot be ranked *a priori*.

2. $VW_{lH} > VW_{hL} \geq VW_{hH}$: $W_{lH}(q_{lH}^*) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}^* > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{hH}(q_{hH}^*) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2}{\alpha_{hH}}\Delta\theta_2 q_{hH}^*$

   $VW_{lH} > VW_{hH} > VW_{hL}$: $W_{lH}(q_{lH}^*) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}^* > W_{hH}(q_{hH}^*) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}^* > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1$

   The buyer would like to increase x_{lH} at the expense of x_{hL}. Doing this while keeping $U_{hL,hH} = U_{hL,lH}$ requires that we adjust the q's and thus λ_2. Specifically, if $\Delta\theta_1 - \Delta\theta_2 q_{lH}^* > 0$, we need to decrease λ_2, otherwise, we need to increase it. In both cases, VW_{lH} goes down and VW_{hH} goes up. This process continues until either a new IC constraint binds or the relative ranking of the virtual welfare changes. Since $x_{lH} > x_{lH}^{FB} > x_{hH}$, the only IC constraint to worry about is $\text{IC}_{lL,lH}$. This gives us three cases to consider depending on which event happens first:

   (a) $VW_{lH} = VW_{hL} \geq VW_{hH}$: We have reached the solution: $x_{lL} = x_{lL}^{FB}$, $x_{hH} = x_{hH}^{FB}$ and $x_{hL}^{FB} > x_{hL} > x_{lH} > x_{lH}^{FB}$ with $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL}) = 1 - \alpha_{hH}^N$, $q_{lL} = q_{hL} = \underline{q}$ and q_{lH} and q_{hH} determined by the value of λ_2 that equates $VW_{hH} = VW_{lH}$. This corresponds to **Solution 1.2.b**.

   (b) $VW_{lH} = VW_{hH} > VW_{hL}$: Note that this means that $q_{lH} < q_{hH}$ and $\Delta\theta_1 - \Delta\theta_2 q_{lH} < 0$ since $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}^{FB}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$. The buyer continues to increase his expected utility by decreasing x_{hL}, this time, to the benefit of both x_{lH} and x_{hH}, doing so while keeping $VW_{lH} = VW_{hH}$ and $U_{hL,lL} = U_{hL,lH}$. Thus λ_2 is fixed and so are q_{lH} and q_{hH}. Therefore $x_{lH} > x_{hH}$. This process continues until $x_{hL} = x_{lH}$ at which point $\text{IC}_{lL,lH}$ starts binding. From then on, the virtual welfares are those defined in (24) - (26). (note that $\lambda_2 = \alpha_{hL} + \lambda_5 - \lambda_3$). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in scenario 1, 2(b) above ($VW_{hH} > VW_{lH} = VW_{hL}$), and the proof thus proceeds along the same lines. [**Solution 2.1.d or 2.1.e**]

(c) $x_{hL} = x_{lH}$, i.e. $\text{IC}_{lL,lH}$ starts binding. From then on, $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding. The virtual welfares are those defined in (24) - (26). (note that $\lambda_2 = \alpha_{hL}+\lambda_5-\lambda_3$). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. If $VW_{lH} = VW_{hL} > VW_{hH}$, we have reached the solution: $x_{lL} = x_{lL}^{FB}$, $x_{hL}^{FB} > x_{lH} = x_{hL} = \overline{x} > x_{lH}^{FB}$, $x_{hH} = x_{hH}^{FB}$, q_{lH}, $q_{hH} < \overline{q}$ and $q_{lL} = q_{hL} = \overline{q}$. All IC constraints are satisfied and the q's and x's are optimal given the resulting virtual welfares. [**Solution 1.2.c**]

If $VW_{lH} = VW_{hL} < VW_{hH}$, we can conclude that $q_{lH} < q_{hH}$ and $\Delta\theta_1 - \Delta\theta_2 q_{lH} < 0$ since $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}^{FB}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$. We are thus in the same situation as in scenario 1, 2(b) above. [**Solution 2.1.d or 2.1.e**]

3. $VW_{hL} > VW_{hH} > VW_{lH} : W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 > W_{hH}(q_{hH}^*) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}^* > W_{lH}(q_{lH}^*) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}^*$.

   (Note that this implies $q_{lH}^* < q_{hH}^*$ and $\Delta\theta_1 - \Delta\theta_2 q_{lH}^* < 0$ given that $x_{lH}^{FB}[W_{lH}(q_{lH}^*) - W_{hL}(q_{lH}^*)]$ $= x_{hH}^{FB}[W_{lH}(q_{hH}^*) - W_{hL}(q_{hH}^*)]$). The buyer wants to increase x_{hH} at the expense of x_{lH}. This requires adjusting λ_2 to maintain the equality $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$. Specifically, λ_2 decreases, q_{lH} increases and q_{hH} decreases, until $VW_{hH} = VW_{lH}$. This occurs at $x_{lH} > x_{hH}$. Indeed, at $x_{lH} = x_{hH}$, $q_{lH} = q_{hH}$ thus $\frac{\alpha_{hL}+\alpha_{lL}-\lambda_2}{\alpha_{hH}}\Delta\theta_2 q_{hH} = \frac{\lambda_2}{\alpha_{lH}}\Delta\theta_2 q_{lH}$ implying that $VW_{hH} < VW_{lH}$. The solution is thus $x_{lL} = x_{lL}^{FB}, x_{hL} = x_{hL}^{FB}$ and $x_{lH}^{FB} > x_{lH} > x_{hH} > x_{hH}^{FB}$ and $q_{lH} < q_{lH} < q_{hH} < \overline{q}$. This corresponds to **solution 2.1.c**

4. $VW_{hH} \geq VW_{lH} \geq VW_{hL} : W_{hH}(q_{hH}^*) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}^* \geq W_{lH}(q_{lH}^*) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}^* \geq W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1$

   Note that this implies that $q_{lH}^* < q_{hH}^*$ and $\Delta\theta_1 - \Delta\theta_2 q_{lH}^* < 0$. Define $\lambda_2^{**} \in (0, \lambda_2^*)$ such that

$$\max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\alpha_{lH}+\lambda_2^{**}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^{**}}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} = \max_{q_{lH}}\{W_{lH}(q_{lH}) + \frac{\lambda_2^{**}}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^{**}}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} \quad (28)$$

This implies $q^*_{lH} < q_{lH} < q_{hH} < q^*_{hH}$ and $VW_{lH} = VW_{hH} > VW_{hL}$.

$>$From there, the buyer can increase his expected utility by increasing x_{hH} and x_{lH} at the cost of x_{hL}. He does so while keeping $\lambda_2 = \lambda_2^{**}$ so that $VW_{hH} = VW_{lH}$. The exact way in which x_{hH} and x_{lH} are increased is pinned down by $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$. This process does not affect the virtual welfare, until $x_{hL} = x_{lH}$ at which point $\text{IC}_{lL,lH}$ starts binding. We are now in a situation where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding and $VW_{hL} < VW_{hH} = VW_{lH}$. From then on, the virtual welfares are those defined in (24) - (26) (note that $\lambda_2 = \alpha_{hL} + \lambda_5 - \lambda_3$). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in 2(b) above. [**Solution 2.1.d or 2.1.e**]

5. $VW_{hH} > VW_{hL} > VW_{lH}$: $W_{hH}(q^*_{hH}) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q^*_{hH} > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 > W_{lH}(q^*_{lH}) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q^*_{lH}$

   We are again in a situation where $q^*_{lH} < q^*_{hH}$ and $\Delta\theta_1 - \Delta\theta_2 q^*_{hH} < 0$. The buyer would like to increase x_{hH} at the expense of x_{lH}. Doing so while keeping $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$ requires an adjustment in λ_2, leading to VW_{lH} decreasing and VW_{hH} increasing. This process continues until we reach λ_2^{**} which corresponds to $VW_{lH} = VW_{hH}$ (as defined in (28)). Since $\Delta\theta_1 - \Delta\theta_2 q^*_{hH} < 0$, the corresponding qualities and x's are such that $q^*_{lH} < q_{lH} < q_{hH} < q^*_{hH}$ and $x_{hH} < x_{lH}$.

   We now need to distinguish two cases depending whether $VW_{hL} > VW_{lH} = VW_{hH}$ or $VW_{lH} = VW_{hH} > VW_{hL}$.

   (a) $VW_{hL} > VW_{lH} = VW_{hH}$: Then we have reached the solution: $x_{lL} = x^{FB}_{lL}$, $x_{hL} = x^{FB}_{hL}$ and $x^{FB}_{lH} > x_{lH} > x_{hH} > x^{FB}_{hH}$, $q_{lL} = q_{hL} = \underline{q}$ and $q^*_{lH} < q_{lH} < q_{hH} < q^*_{hH}$ as defined by (28). This corresponds to **Solution 2.1.c.**

   (b) $VW_{lH} = VW_{hH} > VW_{hL}$: the buyer further increases his expected utility by increases x_{lH} and x_{hH} at the expense of x_{hL} while keeping $VW_{hH} = VW_{lH}$ (that is keeping λ_2 and the q's fixed) and $U_{hL,lH} = U_{hL,hH}$ (thus $x_{hH} < x_{lH}$). This process does not affect the virtual welfares until $x_{hL} = x_{lH}$ and $\text{IC}_{lL,lH}$ starts binding. We are now in a situation where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding and $VW_{hL} < VW_{hH} = VW_{lH}$. From then on, the virtual welfares are those defined in (24) - (26) (note that $\lambda_2 = \alpha_{hL} + \lambda_5 - \lambda_3$). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in 2(b) above, and the proof thus proceeds along the same lines. [**Solution 2.1.d or 2.1.e**]